Published by Institute of Physics Publishing for SISSA

Received: *June 12, 2007*
Revised: *August 21, 2007*
Accepted: *November 28, 2007*
Published: *December 5, 2007*

On phases of generic toric singularities

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Abstract: We systematically study the phases of generic toric singularities, using methods initiated in hep-th/0612046. These correspond to large volume limits of certain Gauged Linear Sigma Models. We show that complete information about the classical regimes of generic $U(1)^r$ GLSMs can be obtained by studying the GLSM Lagrangian, appropriately modified in the different phases of the theory. We use this to study some aspects of $L^{a,b,c}$ spaces and their non-supersymmetric counterparts.

Keywords: Conformal Field Models in String Theory, Tachyon Condensation.

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Contents

1. Introduction

In the last few years, there has been a lot of effort to understand the dynamics of space-times with non-trivial geometries in the framework of string theory. Several deep and fundamental insights have been obtained in the course of the study, which have remarkably contributed to our understanding of the underlying mathematical structure of singular spaces. The central tool in this study has been Witten's Gauged Linear Sigma Model [1] (called GLSM in the sequel) a two dimensional $U(1)^r$ (world sheet) field theory, with $(2,2)$ supersymmetry. It has by now been realised that the classical limits of the GLSM provide a very powerful tool in the analysis of stringy dynamics of non-trivial geometries, especially when these break space-time supersymmetry. A $U(1)^r$ GLSM describing a singular space has r Fayet-Iliopoulos (FI) parameters, which, for space-time non-supersymmetric theories, are one loop renormalized. Tracking the flow of the GLSM between its various classical limits, in terms of the RG flow of the FI parameters gives us information about the different phases of the theory. Indeed, a full understanding of these phases is essential in order to completely specify string theory on non-trivial backgrounds.

Consider, e.g. a $U(1)$ GLSM of four chiral fields, with charges

$$Q = (Q_1, Q_2, Q_3, -Q_4) \tag{1.1}$$

When $\sum_i Q_i \neq 0$, the GLSM describes a non-supersymmetric orbifold via the D-term equation

$$\sum_i Q_i|\phi_i|^2 + r = 0 \tag{1.2}$$

modulo the $U(1)$ identification. Varying the Fayet-Iliopoulos parameter r then determines the behaviour of the model at different points in the moduli space and gives us information about the various possible decays of this unstable orbifold under localised closed string tachyon condensation. In order to fully understand stringy dynamics in this model, one needs to specify the full set of possible Fayet-Iliopoulos parameters in the theory (thereby enlarging the charge matrix) and in general this leads to a rich phase structure. In particular, it has been shown that one can recover the complete set of D-brane charges in these theories by considering the Coulomb branch of the GLSM as well.

Apart from its utility in studying generic orbifold singularities, the GLSM has also been the central tool in the recent advances in our understanding of the extension of Maldacena's AdS/CFT correspondence [2], involving $N = 4$ Super Yang Mills theory, to less supersymmetric situations. According to its original formulation, the AdS/CFT correspondence states that type IIB string theory on $AdS_5 \times S^5$, with appropriately chosen R-R five form flux on the S^5 is dual to large N $N = 4$ Super Yang-Mills theory. This duality has been refined since its inception to include more realistic situations with less supersymmetry, and we now know that type IIB string theory on $AdS_5 \times Y^5$, where Y^5 is a Sasaki-Einstein manifold (i.e a manifold whose metric cone is Calabi-Yau), with appropriate five form flux turned on, is dual to a four dimensional $N = 1$ superconformal field theory (see, e.g [3]). Few explicit examples of Sasaki Einstein manifolds were known till a few years back, when a major breakthrough was achieved in [4], where an infinite class of explicit Sasaki Einstein metrics with topology $S^2 \times S^3$ were constructed.

Much work has followed since then, and the most general family of metrics that have the topology of $S^2 \times S^3$ is denoted by $L^{a,b,c}$, with a, b, c being three positive integers. In the special case when $a = p - q,\ b = p + q,\ c = p$, the $L^{a,b,c}$ metrics reduce to the family of $Y^{p,q}$ metrics (see, e.g [4]). As is well known, the dual $N = 1$, $d = 4$ SCFT in these cases naturally arises as the worldvolume low energy theory of a stack of D-3 branes probing a Calabi-Yau singularity, and residing at the tip of the singular Calabi-Yau cone. These are particularly simple yet illustrative examples, since the Calabi-Yau singularity is a toric variety. The toric description of the $Y^{p,q}$ class of metrics was provided in [5], using which the dual gauge theories were constructed in [6]. Further work [7] has illustrated the GLSM approach to the more general $L^{a,b,c}$ spaces, and the main ingredient in the story is that the $L^{a,b,c}$ toric singularities arise as vacua of GLSMs, with charge matrices of the form[1]

$$Q = (Q_1, Q_2, -Q_3, -Q_4) \tag{1.3}$$

where the Q_is are positive (coprime) integers. This is in distinction to the charge matrix in eq. (1.1) where three of the charges have the same sign. In applications to the AdS/CFT correspondence, the charges are chosen such that they sum to zero, in order to satisfy the Calabi-Yau condition, but in general, this need not be true. When $\sum_i Q_i \neq 0$, the GLSM describes an orbifold of the conifold singularity [8], and results indicate that these might have, in certain regions of moduli space, stable $L^{a,b,c}$ singularities. In other words, as in case of orbifold theories, the singularity in question is unstable and decays to a stable singularity in the sense of the RG. Given the importance of generic $L^{a,b,c}$ spaces, it is important to understand the full phase structure of the GLSMs that may contain the latter in its phases. For example, one can ask if these $L^{a,b,c}$ spaces or their $Y^{p,q}$ cousins can be thought of as

[1]The $L^{a,b,c}$ space is originally defined as the singular space corresponding to the U(1) GLSM of charges $(a, -c, b, -d)$ with $a + b - c - d = 0$ to preserve space-time supersymmetry.

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arising in the moduli space of higher dimensional Calabi-Yau singularities. This might help in a generalised approach to the study of such singular spaces. We will have more to say about this later in this paper.

The phase structure mentioned above can be studied conveniently by constructing the GLSM Lagrangian in its most general form (i.e with all the possible Fayet-Iliopoulos parameters turned on), and then tuning the various Fayet-Iliopoulos parameters of the theory to understand the various phases. The systematic procedure to do this was initiated in [9]. In that paper, the generic $\mathrm{U}(1)^r$ GLSM Lagrangian was constructed in the relevant non-linear sigma model (NLSM) limit, and it was shown how the various phases of the GLSM corresponding to orbifold singularities can be studied in a very general fashion, by providing vevs to certain fields of the GLSM in accordance with the D-term equations of the model under consideration. Although the cases studied in [9] were for orbifold singularities, these can be tuned to study more general GLSMs, like the supersymmetric examples considered in [5, 7], or their non-supersymmetric counterparts [8]. Another aspect of the results in [9] is that using the Lagrangian formulation of the GLSM in the NLSM limit, it is possible to study the behaviour of D-branes in various phases of the GLSM. In its simplest form, this problem reduces to constructing appropriate D-brane boundary conditions in the GLSM [10, 11] and then continuing these appropriately to the different regions of the GLSM moduli space.

The purpose of the present paper is to use and extend the ideas developed in [9] to analyse, in most general terms, the phase structure of generic GLSMs corresponding to unstable spaces. We will show in the course of this paper that the phases of generic GLSMs (corresponding to orbifold spaces or otherwise) can be analysed in a fully algebraic approach, which makes our methods computationally simpler than other existing techniques. We work out several examples as an illustration of our approach. The organisation of the paper is as follows. In section 2, we review some basic results on the GLSM Lagrangians that were obtained in [9]. In section 3, which is the main part of the paper, we use this Lagrangian formulation to extend the analysis of [9], to analyse the phases of GLSMs describing arbitrary toric singularities. This section is divided into two parts. In the first, we study the behaviour of the world sheet gauge field in different phases of arbitrary GLSMs, which is interesting from the point of view of the behaviour of D-branes in the same. In the second part, we analyse in details the sigma model metrics arising in phases of the GLSM. Finally, section 4 concludes with some discussions, and possible extensions of our work.

2. The GLSM lagrangian and singular spaces

In this section, we briefly review the results of [9] in analysing the phases of GLSMs corresponding to generic orbifold singularities. This section is review material, and is meant to set the notations and conventions to be used in the rest of the paper.

Since much of what follows in this paper deals with unstable spaces, let us begin by briefly reviewing the notion of the simplest types of unstable spaces which can be given a toric description, i.e the non-supersymmetric orbifolds of $\mathbb{C}^2$ and their decay properties.

Consider the orbifold of $\mathbb{C}^2$, with action

$$(Z_1, Z_2) \to (\omega Z_1, \omega^p Z_2) \tag{2.1}$$

where Z_1 and Z_2 are coordinates on the $\mathbb{C}^2$, and $\omega = e^{\frac{2\pi i}{n}}$ is the n th root of unity. When $p \neq n-1$, this orbifold action breaks space-time supersymmetry, and introduces tachyons in the closed string spectrum for both Type II and Type 0 strings, that are localised at the tip of the orbifold (and can be interpreted as twisted sector states in the closed string world sheet conformal field theory). A similar action can be written down for $\mathbb{C}^3$ (or $\mathbb{C}$) orbifolds.

An analysis of the condensation of closed string tachyons in this theory then shows that the non-supersymmetric orbifolds decays toward more stable configurations. Whereas for orbifolds of $\mathbb{C}$, the end product of the decay is always flat space, orbifolds of $\mathbb{C}^2$ and $\mathbb{C}^3$ show a much richer structure. The end product of decay of $\mathbb{C}^2$ orbifolds are generally supersymmetric orbifolds of lower rank, for $\mathbb{C}^3$ orbifolds, one might end up reaching a terminal singularity.

The "brane probe" approach of Adams, Polchinski and Silverstein (APS) [12] who first studied these singularities is to use a probe D-brane that has its world volume transverse to the orbifolded directions and is stuck and the orbifold fixed point. The brane probe picture is essentially an open string picture in the substringy regime with localised tachyons, and can be studied by using the gauge theory living on the world volume of the D-brane. In the APS procedure, it is found that by exciting the marginal (or tachyonic deformations) in the theory, one can drive the original orbifold to one of lower rank, and possible tachyonic deformations of the resulting theory takes the system to a final stable supersymmetric configuration. An useful alternative (closed string) approach is to study the $N = (2, 2)$ SCFT of the worldsheet, which is related to Witten's GLSM. [13]. One can construct an appropriate GLSM corresponding to the non-supersymmetric orbifolds, and track the behaviour of the model in the sense of the RG, and this provides us with an alternative description for decays of non-supersymmetric orbifolds via closed string tachyon condensation.

In a related approach to the problem of tachyon condensation, the GLSM Lagrangian and hence the sigma model metrics (with possible multiple U(1) gauge groups) was studied for the non-supersymmetric $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds [14, 15]. To illustrate the idea, let us begin with a brief description of the GLSM. The action for a GLSM with, with an Abelian gauge group U(1) is given by

$$S = \int d^2 z d^4\theta \sum_i \bar{\Phi}_i \Phi_i - \frac{1}{4e^2} \int d^2 z d^4\theta \bar{\Sigma}\Sigma + Re\left[it \int d^2 z d^2\tilde{\theta}\Sigma \right] \tag{2.2}$$

where the Φ_i are chiral superfields, Σ is a twisted chiral superfield, $t = ir + \frac{\theta}{2\pi}$ is a complexified parameter involving the FI parameter r and the two dimensional θ angle. As appropriate in our case, we consider a theory without a superpotential. In general, we will consider GLSMs with multiple U(1) gauge groups, in which case the twisted chiral superfields Σ carries an extra index, along with the gauge coupling and the complex FI parameter.

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In the $e^2 \to \infty$ limit of the GLSM (called the NLSM limit in the sequel), the gauge fields appearing in (2.2) are Lagrange multipliers. It is then possible to obtain the Lagrangian and solve the D-term constraint in the classical limit $|r| \to \infty$ to read off the sigma model metric corresponding to the GLSM [15, 9]. Focusing on the bosonic part of the GLSM action, given by

$$S = -\int d^2 z D_\mu \bar{\phi}_i D^\mu \phi_i \tag{2.3}$$

the Lagrangian can be studied using the D-term constraints,

$$\sum_i Q_i^a |\phi_i|^2 + r_a = 0 \tag{2.4}$$

where ϕ_i are the bosonic components of the Φ_i and Q_i^a denote the charges of the ϕ_i with respect to the ath U(1). Orbifolds of the type $\mathbb{C}^r/\Gamma$, with $r = 1, 2, 3$ can be described by GLSM, with the number of the gauge groups being dictated by the nature of the singularity. In the NLSM limit, the component gauge fields in the model can be calculated and substituted back into the action to get the GLSM Lagrangian entirely in terms of the toric data of the singularity.

It can be shown that the Lagrangian for a GLSM with the m fields ϕ_i, $i = 1, 2, \ldots, m$ with single U(1) gauge group with charges Q_i, $i = 1, 2, \ldots m$ is given by:

$$L = (\partial_\mu \rho_1)^2 + (\partial_\mu \rho_2)^2 + \cdots + (\partial_\mu \rho_m)^2 + \frac{\sum_{i<j} \rho_i^2 \rho_j^2 (Q_i \partial_\mu \theta_j - Q_j \partial_\mu \theta_i)^2}{\sum_j Q_j^2 \rho_j^2} \tag{2.5}$$

Where we have written the complex fields $\phi_i = \rho_i e^{i\theta_i}$. In the classical limits of the GLSM (corresponding to the modulus of the FI parameter being very large), this formula gives the sigma model metric for the singularity $\mathbb{C}^{m-1}/\mathbb{Z}_n$. As we have mentioned before, this corresponds to giving a large vev to any of the fields appearing in the Lagrangian.

Following a similar approach, the Lagrangian for the two parameter GLSM can be constructed. For m fields ϕ_i, $i = 1, 2, \ldots, m$ and two gauge groups $a, b = 1, 2$, the expression for the Lagrangian is:

$$L = L_1 + L_2 \tag{2.6}$$

where

$$L_1 = \sum_i (\partial_\mu \rho_i)^2 \tag{2.7}$$

$$L_2 = \frac{\sum_{[i,j,k]} [\rho_i \rho_j \rho_k \partial_\mu \theta_i (Q_j^b Q_k^a - Q_k^b Q_j^a)]^2}{\sum_{i<j} \rho_i^2 \rho_j^2 (Q_i^b Q_j^a - Q_i^a Q_j^b)} \tag{2.8}$$

Where the symbol $[i, j, k]$ in the summation in the numerator in L_2 denotes *cyclic* combinations of the variables and as before, we have written $\phi_i = \rho_i e^{i\theta_i}$. This expression can be used to study non-cyclic singularities of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ as well, which can not be described by a single parameter GLSM.

The above Lagrangians can be generalized to the case of the general r parameter GLSMs. The general r parameter GLSM Lagrangian can be written as

$$L = L_1 + L_2 \tag{2.9}$$

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where now

$$L_1 = \sum_i (\partial_\mu \rho_i)^2 \tag{2.10}$$

$$L_2 = 6\frac{\sum_{[j_1,j_2,\ldots,j_{r+1}]}[\rho_{j_1}\rho_{j_2}\ldots\rho_{j_r}\partial_\mu(\theta_{j_1}K_{j_2,\ldots,j_r})]^2}{\sum_{j_1<j_2<\ldots<j_r}\rho_{j_1}^2\rho_{j_2}^2\ldots\rho_{j_r}^2[\Delta(j_1,j_2,\ldots,j_r)]^2} \tag{2.11}$$

where i and $j_1, j_2, \ldots, j_{r+1}$ go from $1, 2, \ldots, n$, where n is the total number of scalar fields. $K_{j_2,\ldots,j_r}$ is the $j_1 th$ component of the kernel of the matrix formed by the charges of the j_{r+1} vectors in the numerator of L_2(and hence depends on $j_2, j_3, \ldots, j_{r+1}$), and $\Delta(j_1, j_2, \ldots, j_r)$ is the determinant of the matrix formed by the charge vectors $\rho_{j_1}, \rho_{j_2}, \ldots, \rho_{j_r}$ under the r U(1)s. Again the notation $[j_1, j_2, \ldots, j_{r+1}]$ indicates a cyclic combination of the variables.

The derivation of eqs. (2.5)–(2.11) crucially uses the expressions for the world sheet gauge fields, and we list the results for the same. It will be sufficient for us to write the expression for the gauge fields for the most general case. For a $\mathrm{U}(1)^r$ GLSM with fields ϕ_i having charges $Q_i^{(a)}$ under the ath U(1), a straightforward calculation shows that the expression for the gauge field is given by

$$v_\mu^{(n)} = \frac{\Delta_n}{\Delta} \tag{2.12}$$

where $\mu = 0, 1$, $n = 1, \cdots, r$ and $\Delta = \det$ A, with the matrix A being given by

$$A = \begin{pmatrix} K^{11} & K^{12} & \cdots & K^{1r} \\ \cdots & \cdots & \cdots & \cdots \\ K^{r1} & K^{r2} & \cdots & K^{rr} \end{pmatrix} \tag{2.13}$$

where

$$K^{ab} = \sum_i Q_i^{(a)} Q_i^{(b)} |\phi_i|^2 \tag{2.14}$$

a and b going from $1 \cdots r$. Δ_n is the determinant of the matrix A with the nth column being replaced by the column matrix

$$J = \left(\sum_i Q_i^{(1)} (\mathrm{Im}\bar{\phi}_i \partial_\mu \phi_i), \cdots \sum_i Q_i^{(r)} (\mathrm{Im}\bar{\phi}_i \partial_\mu \phi_i) \right)^T \tag{2.15}$$

where the sum is over all the fields in the theory. A detailed derivation of the above result will not be useful for us. Note that for the special case of $r = 1$, this reduces to the familiar result [15]

$$v_\mu = \frac{\sum_i Q_i \mathrm{Im}(\bar{\phi}_i \partial_\mu \phi_i)}{\sum_i Q_i^2 |\phi_i|^2} \tag{2.16}$$

Having written down the GLSM Lagrangian and the world sheet gauge field in its most general form entirely in terms of the toric data of the orbifold, we can now use this formalism to study the classical phases of generic GLSMs. For a multi-parameter GLSM, these phases are obtained from the Lagrangian by making (the modulus of) some of the fields in the GLSM very large.[2] In [9], this approach was used to study the phases of orbifold GLSMs, and it was shown how non-cyclic orbifolds of $\mathbb{C}^3$, i.e orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$ can also be handled easily in this framework.

In what follows we will mostly present explicit formulae for GLSMs with one or two gauge groups. It should be clear from our discussion in this section that these can be carried over to more complicated examples involving multiple gauge groups.

3. GLSM analysis of generic singular spaces

In this section, we will study the phases of generic GLSMs, extending the analysis of [9], using the Lagrangian formulation developed therein and discussed in the previous section. Whereas our previous study focused on the sigma model metrics in phases of orbifold singularities, we will be more general here. Generically, given an r parameter GLSM with say m fields, we would like to study its classical limits by giving large vevs to an arbitrary number of fields. Before we present the full Lagrangian analysis of the classical phases of generic GLSMs, we will discuss an important subset of our results, the behaviour of the world sheet gauge fields in generic phases of unstable spaces.

3.1 Behaviour of the gauge field in phases of unstable spaces

As is well known, the GLSM, which enjoys $(2,2)$ worldsheet supersymmetry, can be used to construct the spectrum of D-branes on singular spaces. Specifically, one can construct D-branes in the GLSM by considering the open string boundary conditions that preserve (a part of the) supersymmetry, and the behaviour of D-branes in the classical phases of the GLSM is captured by the behaviour of these boundary conditions in the various regions of moduli space. An important ingredient in this study is the behaviour of the world sheet gauge fields. Consider a single parameter GLSM. It can be shown following [10] that in the Higgs branch of the theory, the world sheet gauge field is related to its space-time counterpart as $mv_0 = A_\mu \partial_0 X^\mu$ where m refers to the mth fractional D2- brane. It was pointed out in [15] that by computing the flux of the space-time gauge field (which follows from this identification), one can study the behaviour of D-branes in the classical limits of the GLSM.

For the case of the simplest orbifold $\mathbb{C}/\mathbb{Z}_n$, it can be shown by explicit computation that whereas in the far UV region the flux of the space-time gauge field is localised at the origin, it vanishes in the far IR region, thus rendering the (fractional) D2- branes of the theory indistinguishable [15]. It is thus important to understand the behaviour of the gauge fields in the phases of generic GLSMs, in order to understand D-brane dynamics in these, and in this subsection, we present a preliminary analysis of the same. In what follows, without loss of generality, we will focus on the behaviour of the world sheet gauge field v_0. A detailed description of the behaviour of D-branes would require a careful analysis of the geometry probed by the gauge fields, by writing down the spacetime gauge field, and using

[2]This corresponds to a region of the moduli space where some combination of the Fayet-Iliopoulos parameters being very large.

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the gauge freedom of the unbroken U(1)s to study the gauge invariant angles that will appear in the results. We will however, leave such an analysis for the future, and for the purpose of this paper, present some qualitative features of the behaviour of the worldsheet gauge field.

Before we begin, let us make a few comments. Firstly, note that there is a small subtlety regarding the charges of the chiral fields in our Lagrangian formulation. Consider, e.g. the two parameter GLSM of eqs. (2.8) and (2.11). Namely, in these equations, the square of the charges appear in the denominator, and hence when some of the fields are given very large vevs, it might seem that there is a sign ambiguity in the definition of the charges of the remaining fields in the reduced Lagrangian. However, it is not difficult to convince oneself that there is actually no such ambiguity. It is best to illustrate this with an example. Consider, e.g. the GLSM corresponding to the unstable orbifold $\mathbb{C}^2/\mathbb{Z}_{5(3)}$. The closed string description of this singularity tells us that there are two twisted sectors that participate in the full resolution (corresponding to divisors with intersection numbers -2 and -3, and hence the $\mathrm{U}(1)^2$ charge matrix for this singularity is given by

$$Q = \begin{pmatrix} 1 & 3 & -5 & 0 \\ 2 & 1 & 0 & -5 \end{pmatrix} \tag{3.1}$$

Writing the fields as $\phi_i = \rho_i e^{i\theta_i}$, in the limit that one of the fields, say $|\phi_1| \gg 0$, we substitute this charge matrix in eq. (2.7), to obtain the (relevant part of the) reduced Lagrangian $L = \frac{N}{D}$ where now

$$\begin{aligned} N &= \rho_2^2\rho_3^2\left(-\partial_\mu\theta_3 - 2\partial_\mu\theta_2 + \partial_\mu\theta_1\right)^2 + \rho_2^2\rho_4^2\left(-\partial_\mu\theta_4 + \partial_\mu\theta_2 - 3\partial_\mu\theta_1\right)^2 \\ &\quad + \rho_3^2\rho_4^2\left(2\partial_\mu\theta_4 + \partial_\mu\theta_3 + 5\partial_\mu\theta_1\right)^2 \\ D &= \left[\rho_2^2(1)^2 + \rho_3^2(-2)^2 + \rho_4^2(+1)^2\right] \end{aligned} \tag{3.2}$$

Note that the terms in N correspond to the gauge invariant angles, and we have explicitly indicated the fact that in the denominator, the original signs appearing with the various terms in eq. (2.7) have to be retained (modulo possibly an overall relative sign between the terms). The value of D shows that we now have a reduced charge matrix for the fields ϕ_2, ϕ_3, ϕ_4 with

$$Q = (1, -2, 1) \tag{3.3}$$

which is the GLSM for the supersymmetric orbifold $\mathbb{C}^2/\mathbb{Z}_2$.

In general, in a two parameter GLSM, making one field large (i.e giving it a large vev) will not break the full $\mathrm{U}(1)^2$ symmetry. Consider, e.g. the charge matrix in eq. (3.1). The two D-term constraints coming from this charge matrix is given by

$$\begin{aligned} |\phi_1|^2 + 3|\phi_2|^2 - 5|\phi_3|^2 + r_1 &= 0 \\ 2|\phi_1|^2 + |\phi_2|^2 - 5|\phi_4|^2 + r_2 &= 0 \end{aligned} \tag{3.4}$$

Setting $r_1 \ll 0$, we can solve for $|\phi_1|$ as

$$|\phi_1| = \sqrt{5|\phi_3|^2 - 3|\phi_2|^2 - r_1} \tag{3.5}$$

Now, substituting this value of $|\phi_1|$ in the second of the D-term equations, we see that there is a residual unbroken U(1) with a modified D-term constraint

$$-5|\phi_2|^2 + 10|\phi_3|^2 - 5|\phi_4|^2 + (r_2 - 2r_1) = 0 \tag{3.6}$$

In order to completely break the original $\mathrm{U}(1)^2$, we now need to give a vev to a second field. Hence, the GLSM that we obtain by making one field have a very large vev refers to this residual U(1). Now, the relevant part of the one parameter Lagrangian[3] with this charge matrix is given by $L = N_1/D_1$, where $D_1 = D$ of eq. (3.2) and

$$N_1 = \rho_2^2\rho_3^2\left(\partial_\mu\tilde{\theta}_3 + 2\partial_\mu\tilde{\theta}_2\right)^2 + \rho_2^2\rho_4^2\left(\partial_\mu\tilde{\theta}_4 - \partial_\mu\tilde{\theta}_2\right)^2 + \rho_3^2\rho_4^2\left(-2\partial_\mu\tilde{\theta}_4 - \partial_\mu\tilde{\theta}_3\right)^2 \tag{3.7}$$

where we have denoted the angular variables in the reduced Lagrangian with a tilde. Now, with the identification

$$\tilde{\theta}_2 = \theta_2, \quad \tilde{\theta}_3 = \theta_3 - \theta_1, \quad \tilde{\theta}_4 = \theta_4 + 3\theta_1 \tag{3.8}$$

we see that the two Lagrangians are identical. This analysis tells us that the supersymmetric $\mathbb{C}^2/\mathbb{Z}_2$ orbifold arises as a decay product of the unstable $\mathbb{C}^2/\mathbb{Z}_{5(3)}$ orbifold. Given the generality of our analysis, it should be clear that this can be used to analyse the phases of *any* GLSM with an arbitrary number of gauge groups and arbitrary charges.

Before we proceed, let us point out a simple rule to read off the charges of the resulting GLSM when some of the fields have been integrated out. This will be useful for us in what follows. We find that for a $\mathrm{U}(1)^r$ GLSM, with $r+2$ or $r+3$ fields (according to whether it describes a $\mathbb{C}^2$ or a $\mathbb{C}^3$ singularity), giving vevs to $r-1$ fields, one can read off the charges of the remaining 3 or the remaining 4 fields as follows. In the GLSM charge matrix, we collect the charges of the fields being resolved, an a $r \times r-1$ matrix. Then the redefined charges (upto probably an overall unimportant sign) of the remaining fields can be read off as the determinant of the $r \times r$ matrix formed by augmenting the $r \times r - 1$ matrix of the resolved fields with each of the remaining fields in turn, while keeping the order of the first $r-1$ fields intact.

We now begin our discussion of the behaviour of (worldsheet) gauge fields in phases of generic GLSMs. To illustrate the idea, let us consider the GLSM for the orbifold $\mathbb{C}^2/\mathbb{Z}_{n(k)}$, with the charge matrix $Q = (1, k, -n)$. (The example of the $\mathbb{C}$ orbifold was considered in [15]). The explicit solution for the D-term equation for this model, in the IR, is

$$\phi_1 = \rho_1 e^{i\theta_1}, \; \phi_2 = \sqrt{\frac{|r| + n\rho_3^2 - \rho_1^2}{k}}e^{i\theta_2}, \; \phi_3 = \rho_3 e^{i\theta_3} \tag{3.9}$$

this leads to the world-sheet gauge field

$$v_0 = \frac{\rho_1^2\partial_0\theta_1 - n\rho_3^2\partial_0\theta_3 + \left(n\rho_3^2 - \rho_1^2 + |r|\right)\partial_0\theta_2}{\rho_1^2 + n^2\rho_3^2 + k\left(n\rho_3^2 - \rho_1^2 + |r|\right)} \tag{3.10}$$

[3] In the L_1 component in eq. (2.7) or eq. (2.10), the field that has been made large drops out.

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This is easily seen to be the the gauge field v_0 in the UV, for the GLSM with the charge matrix $(1, 2k-n, -k)$, after we enforce the condition that in the latter case, all integers are defined modulo k. A similar analysis holds for the space-time gauge field as well.

From the analysis that we have presented in the previous section, it should be clear that the treatment of the last paragraph can proceed without the explicit solution of the D-term equations. It is easy to see this. Consider, e.g the GLSM for two fields (ϕ_1, ϕ_2), with charges

$$Q = (1, -n) \tag{3.11}$$

which, in the UV limit is the orbifold $\mathbb{C}/\mathbb{Z}_n$. We will consider the world sheet gauge field v_0, which is given by

$$v_0 = \frac{\rho_1^2 \partial_0 \theta_1 - n\rho_2^2 \partial_0 \theta_2}{\rho_1^2 + n^2 \rho_2^2} \tag{3.12}$$

where, as before, we have denoted the complex fields $\phi_i = \rho_i e^{i\theta_i}$. In the limit $\rho_2 \gg 0$, we recover $v_0 = -\partial_0 \left(\frac{\theta_2}{n}\right)$ and in the IR, when $\rho_1 \gg 0$, we have $v_0 = \partial_0 \theta_1$. This implies that the flux of the space-time gauge field vanishes in the IR, and is localised at the conical defect in the UV. The same conclusion can be reached by explicitly solving for v_0 in terms of the FI parameter of the theory [15].

An entirely similar analysis can be done for orbifolds of the form $\mathbb{C}^2/\mathbb{Z}_{n(k)}$. Let us first concentrate on the single parameter GLSM description of this orbifold, which consists of three fields $\phi_i,\ i = 1, \cdots, 3$ that we denote as $\phi_i = \rho_i e^{i\theta_i}$. The have $\mathrm{U}(1)$ charges denoted by $Q = (1, k, -n)$. For this model, the world sheet gauge field v_0 is solved as

$$v_0 = \frac{\rho_1^2 \partial_0 \theta_1 + k\rho_2^2 \partial_0 \theta_2 - n\rho_3^2 \partial_0 \theta_3}{\rho_1^2 + k^2 \rho_2^2 + n^2 \rho_3^2} \tag{3.13}$$

In the UV limit (which corresponds to setting $\rho_3 \gg 0$), we see that the flux of the spacetime gauge field is localised at the origin of a cone with conical deficit $\frac{2\pi}{n}$. In the IR, we can either take ρ_1 or ρ_2 to be very large (as is obvious from the D-term equation for this model). In the former case, the flux is seen to be zero in the IR, but it is localised at the origin of a cone with the conical deficit being $\frac{2\pi}{k}$ when we consider the latter case. This shows that under the RG evolution of the FI parameter, the gauge field flux gets trapped at the tip of a cone with a greater conical deficit. The process stops when one reaches a supersymmetric configuration, and there is no further RG flow of the FI parameter.

Let us point out that for generic r parameter GLSMs in the classical limit, making some fields acquire a large vev is equivalent to reducing the number of $\mathrm{U}(1)$ gauge groups, from our discussion of section 2. It can be shown that in this case, some of the gauge fields become appropriately identified, upon identification of the gauge invariant angles. E.g for two parameter theories, by giving a large vev to one of the fields (which effectively reduces the theory to a $\mathrm{U}(1)$ GLSM, the gauge fields $v_0^{(1)}$ and $v_0^{(2)}$ can be identified as the single gauge field for the reduced theory. We will illustrate this with the example of the unstable orbifold $\mathbb{C}^2/\mathbb{Z}_{5(3)}$, which has two twisted sectors, and hence is described by the $\mathrm{U}(1)^2$ GLSM of four fields $\phi_i, i = 1 \cdots 4$, with the charge matrix given eq. (3.1).

The world sheet gauge fields can be calculated in this model by using the formulae in eqs. (2.12)–(2.15). We will concentrate on the gauge field v_0. In the phase of this theory where $|\phi_1|$ is very large, we obtain the GLSM for the supersymmetric orbifold $\mathbb{C}^2/\mathbb{Z}_2$ with the charge matrix $(1,-2,1)$ (and zero FI parameter), and the expressions for the gauge field $v_0^{(1)}$ is

$$v_0^{(1)} = \frac{\rho_2^2\partial_0\tilde{\theta}_2 - 2\rho_3^2\partial_0\tilde{\theta}_3 + \rho_4^2\partial_0\tilde{\theta}_4}{\rho_2^2 + 4\rho_3^2 + \rho_4^2} \tag{3.14}$$

where the angles turn out to be gauge invariant with respect to the second U(1) of eq. (3.1),

$$\tilde{\theta}_2 = (2\theta_2 - \theta_1)\,, \quad \tilde{\theta}_3 = (2\theta_3)\,, \quad \tilde{\theta}_4 = (2\theta_4 + \theta_1) \tag{3.15}$$

and we have used the fact that the integers appearing in the expressions above are defined modulo 2. A similar calculation for the field $v_0^{(2)}$ now shows that it is indeed identical to $v_0^{(1)}$ above, with a different choice of the gauge invariant angles, now with respect to the second U(1). This is therefore (the zeroth component of) the gauge field of the U(1) left unbroken by our choice of vevs.Now, once can consider any other phase where another field is very large, and it is seen that now turning on very large vevs for ϕ_2 and ϕ_4 result in the dilution of the flux of the space-time gauge fields, whereas giving a large vev to ϕ_2 results in the flux being trapped at the tip of a cone with a conical deficit of $\frac{\pi}{2}$.

Next, let us consider orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_m$. This case is interesting, as it has, among its phases, unstable $L^{a,b,c}$ or $Y^{p,q}$ toric singularities [5]. Let us see how this works out in this framework. As a concrete example, we take the space-time supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. The toric data (and hence the charge matrix for the GLSM) for this model can be worked out by using the standard closed string prescription of constructing the twisted sectors of the theory, and then restoring integrality in the $\mathbb{Z}^{\oplus 3}$ formed by the twisted sector R charges and the generators of the lattice (see, e.g [17, 18, 9]). Choosing the two $\mathbb{Z}_3$ actions as

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega Z^1, \omega^2 Z^2, Z^3\right) \\ g_2 &: \left(Z^1, Z^2, Z^3\right) \to \left(\omega Z^1, Z^2, \omega^2 Z^3\right) \end{aligned} \tag{3.16}$$

($\omega = e^{\frac{2\pi i}{3}}$ and including the fractional points corresponding to the action of $g_1.g_2$, we obtain the toric data for the $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold [9]

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 2 & -1 & 2 & 3 & 1 & -2 & 4 \\ 0 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 & 2 & -2 & -4 & -1 & 3 & -6 \end{pmatrix} \tag{3.17}$$

The charge matrix for the $\mathrm{U}(1)^7$ GLSM of 10 fields $\phi_i, i = 1, \cdots 10$ is the kernel of $\mathcal{T}$, and is given (in a particular basis) by

$$\mathcal{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & -3 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & -2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -2 & 3 \end{pmatrix} \tag{3.18}$$

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Now, we consider the phase of this GLSM in which some of the fields are given very large vevs. If we give make $|\phi_i|, i = 1 \cdots 6$ very large, the resulting gauge fields can been seen to correspond to that of a $\mathrm{U}(1)$ GLSM with charge matrix:

$$Q = (1, 3, -2, -2) \tag{3.19}$$

With zero FI parameter, this is the space $Y^{2,1}$ whose toric diagram can be embedded in that of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ [5]. We can also read off the charges of the remaining fields under different choices of the unbroken $\mathrm{U}(1)$. E.g if we give the fields $\phi_1, \cdots, \phi_5$ and ϕ_7, we obtain GLSM of the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ singularity, while giving vevs to $\phi_1, \cdots, \phi_5$ and ϕ_8 gives the unbroken $\mathrm{U}(1)$ corresponding to the space $L^{1,2,1}$, or the suspended pinch point singularity.

It is simple to generalise the above to the case of product orbifolds that act asymmetrically on the coordinates of $\mathbb{C}^3$, e.g the supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_5$, with the orbifolding action for the two groups being

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3\right) \to \left(-Z^1, -Z^2, Z^3\right) \\ g_2 &: \left(Z^1, Z^2, Z^3\right) \to \left(Z^1, \omega Z^2, \omega^{-1} Z^3\right) \end{aligned} \tag{3.20}$$

From the twisted sectors of the theory, we find that this orbifold is described by a $\mathrm{U}(1)^7$ GLSM of 10 fields. The behaviour of the world sheet gauge field can be easily analysed in various phases of this theory, and it can be checked that the suspended pinch point singularity arises in this theory as well, i.e its toric diagram can be embedded in the singularity $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_5$.

Our formulation can be carried over with ease to the unstable non-cyclic singularities of $\mathbb{C}^3$ as well. We will be brief here, and consider the simple example of the space-time non-supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_{5(2)} \times \mathbb{Z}_{5(2)}$ where the action of the $\mathbb{Z}_5$ introduces tachyons in the closed string spectrum. The GLSM for this orbifold can be calculated using its toric data [9] is given, in a particular basis, as the $\mathrm{U}(1)^5$ theory of eight fields $\phi_i, i = 1 \cdots 8$,

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & -1 & -5 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 & -3 \end{pmatrix} \tag{3.21}$$

We find that giving large vevs to the first four fields ϕ_i results in the GLSM of remaining four fields with charges $(1, 2, -1, -3)$, while the phase in which the fields ϕ_1, ϕ_2, ϕ_3 and ϕ_5 have a large vev is the supersymmetric conifold. The gauge fields can be computed as before, and various lengthy combinations of gauge invariant angles occur and we will not present the explicit expressions here. In the former case, as before, making a fifth field very large may result in the flux of the space-time gauge field strength being completely diluted (i.e the resulting space is flat) or it might correspond to a conical deficit corresponding to an orbifold of $\mathbb{C}^3$.

Finally, we comment briefly on the behaviour of the gauge field in orbifolds of $\mathbb{C}^4$. This has a substantially richer structure than our previous examples. However, one needs to be careful here, since many members of this class orbifolds of $\mathbb{C}^4$ do not admit a crepant resolution. We leave a detailed study of these (space-time supersymmetric and non-supersymmetric) orbifolds for the future, and for the purpose of this paper, we will consider a few supersymmetric examples, which will qualitatively illustrate the broad applicability of our results.

Our first example is that of the orbifold $\mathbb{C}^4/\mathbb{Z}_{10}$ with the orbifolding action

$$\left(Z^1, Z^2, Z^3, Z^4\right) \rightarrow \left(\omega Z^1, \omega Z^2, \omega Z^3, \omega^7 Z^4\right) \tag{3.22}$$

with ω being the 10th root of unity. This orbifold does admit a crepant resolution [19]. The toric data for this orbifold can be calculated by standard means (e.g by restoring integrality in the fractional $\mathbb{Z}^{\oplus 4}$ lattice obtained from the twisted sector R charges) and it can be shown that this orbifold is described by a $\mathrm{U}(1)^3$ GLSM of 7 fields, and by making some of the fields large as before, one can read off the behaviour of the world sheet gauge field in different phases of this theory. E.g by making three of the seven fields of this theory very large (so as to retain a single unbroken $\mathrm{U}(1)$, we recover the supersymmetric $\mathbb{Z}_4$ orbifold of $\mathbb{C}^4$ and the $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$ theory.

Product orbifolds of $\mathbb{C}^4$ are also easy to study. As a simple example, consider the orbifold $\mathbb{C}^4/\mathbb{Z}_5 \times \mathbb{Z}_5$ where the orbifolding action

$$\begin{aligned} g_1 &: \left(Z^1, Z^2, Z^3, Z^4\right) \rightarrow \left(\omega Z^1, \omega Z^2, \omega^3 Z^3, Z^4\right) \\ g_2 &: \left(Z^1, Z^2, Z^3, Z^4\right) \rightarrow \left(\omega Z^1, \omega Z^2, Z^3, \omega^3 Z^4\right) \end{aligned} \tag{3.23}$$

with $\omega = e^{\frac{2\pi i}{5}}$ makes the orbifold space-time supersymmetric. In this case, the GLSM charge matrix in a particular basis is given by (the twisted sectors under the action of $g_1.g_2$ are all irrelevant)

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & -5 & -2 & 6 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & -1 & 3 \end{pmatrix} \tag{3.24}$$

Labeling the fields as before as $\phi_i, i = 1, \cdots, 8$, and e.g giving large vevs to the fields ϕ_2, ϕ_3, ϕ_4, we unbroken $\mathrm{U}(1)$ corresponds to the singular space $L^{1,3,1}$ with the GLSM charge matrix $(1, -1, 3, -3)$.

We have also studied the slightly more complicated example of the supersymmetric orbifold $\mathbb{C}^4/\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, extending the action for the $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold presented before. By including the various twisted sectors of the theory, we find a $\mathrm{U}(1)^{16}$ GLSM of 20 fields. We find that by giving large vevs to 15 of the fields, so as to again retain a single $\mathrm{U}(1)$, we obtain the supersymmetric conifold in many regions of moduli space. It would be interesting to explore this further, and to understand the phases of $\mathbb{C}^4$ orbifolds with terminal singularities, using our methods.

3.2 Decay of generic unstable spaces

We will now present our analysis of singular spaces corresponding to GLSMs with charges

$$Q = (Q_1, Q_2, -Q_3, -Q_4) \tag{3.25}$$

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with the Q_i being positive integers, using the Lagrangian analysis discussed in section 2, and calculate the explicit sigma model metrics for the same. As discussed in [6], this is the most general charge configuration for a U(1) GLSM with four fields which does not describe an orbifold singularity. This is because all the charges have been taken to be non-zero, hence either two of them or three of them have the same sign, but the latter are simply orbifolds of $\mathbb{C}^3$ so for our purposes, it is enough to begin with the charge matrix of eq. (3.25). For the Calabi-Yau condition to be satisfied, one requires that $\sum_i Q_i = 0$, but we will not put such a restriction here, and would consider the general case where $\sum_i Q_i \neq 0$. The Lagrangian corresponding to the infinite gauge coupling limit of the GLSM, with the D-term constraint being

$$Q_1|\phi_1|^2 + Q_2|\phi_2|^2 - Q_3|\phi_3|^2 - Q_4|\phi_4|^2 + r = 0 \tag{3.26}$$

is given by setting $m = 4$ in eq. (2.5),

$$L = (\partial_\mu \rho_1)^2 + \cdots + (\partial_\mu \rho_4)^2 + \frac{\sum_{i,j=1,\cdots 4,i<j} \rho_i^2 \rho_j^2 (Q_i \partial_\mu \theta_j - Q_j \partial_\mu \theta_i)^2}{\sum_j Q_j^2 \rho_j^2} \tag{3.27}$$

where the fields with charge Q_i have been written as $\phi_i = \rho_i e^{i\theta_i}$.

We now look at the classical limits of this GLSM. This can be done by setting the (magnitude of the) Fayet-Iliopoulos parameter to be very large. Specifically, setting r to be very large positive, we see that (the modulus of) either ϕ_3 or ϕ_4 has to be made very large. If we choose ϕ_4 to be very large, we can solve for the fields in the classical limit as

$$\phi_1 = \rho_1 e^{i\theta_1}, \quad \phi_2 = \rho_2 e^{i\theta_2}, \quad \phi_3 = \rho_3 e^{i\theta_3}, \quad \phi_4 = \sqrt{\frac{Q_1\rho_1^2 + Q_2\rho_2^2 - Q_3\rho_3^2 + r}{Q_4}} e^{i\theta_4} \tag{3.28}$$

Substituting these values in the Lagrangian yields

$$L = \sum_{i=1}^{3} (\partial_\mu \rho_i)^2 + \rho_1^2 d\tilde{\theta}_1^2 + \rho_2^2 d\tilde{\theta}_2^2 + \rho_3^2 d\tilde{\theta}_3^2 \tag{3.29}$$

where

$$\tilde{\theta}_1 = \theta_1 + \frac{Q_1}{Q_4}\theta_4, \quad \tilde{\theta}_2 = \theta_2 + \frac{Q_2}{Q_4}\theta_4, \quad \tilde{\theta}_3 = \theta_3 - \frac{Q_3}{Q_4}\theta_4 \tag{3.30}$$

This can be recognised as the Lagrangian corresponding to the orbifold GLSM with charges

$$Q = (Q_1, Q_2, pQ_4 - Q_3, -Q_4) \tag{3.31}$$

where p is the smallest positive integer that makes $pQ_4 - Q_3$ a positive number. Similarly, if we set ϕ_3 to be very large, we obtain the Lagrangian corresponding to the classical limit of the GLSM with charges

$$Q = \left(Q_1, Q_2, p'Q_3 - Q_4, -Q_3\right) \tag{3.32}$$

where, as before we have introduced an integer p' to make the third entry in the above equation positive.

The analysis for $r \ll 0$ can be carried out in exactly the same way, and the corresponding orbifold singularities have ranks Q_1 and Q_2. This shows that the GLSM with charges given in eq. (3.25) contain orbifold singularities in their classical limits (This conclusion has been reached by other methods in [8]). The full phase structure of the GLSM can thus be studied by including the additional blow up modes that follow from these orbifolds. Let us see if we can substantiate this. Consider, e.g. the simpler class of the supersymmetric $Y^{p,q}$ singularities, described by the GLSM with charge matrix

$$Q = (p-q, p+q, -p, -p) \tag{3.33}$$

The D-term constraint in this case reads

$$(p-q)\,|\phi_1|^2 + (p+q)\,|\phi_2|^2 - p\left(|\phi_3|^2 + |\phi_4|^2\right) + r = 0 \tag{3.34}$$

in the classical limits, we can solve the D-term constraint as before. Consider e.g. the limit $r \gg 0$. In this case, we can choose to set the magnitude of ϕ_4 to be very large. Substituting the result in eq. (3.27) we see that the resulting Lagrangian has the same form as that in eqs. (3.29) and (3.30), excepting that now the coordinate corresponding to ρ_3 (and θ_3) are unorbifolded, leading to the fact that in this limit we actually have a supersymmetric $\mathbb{C}^2/\mathbb{Z}_p$ singularity. A similar result is obtained on setting $\phi_3 \gg 0$ wherein we recover the same singularity. In the other limit, i.e when $r \ll 0$, we recover two supersymmetric $\mathbb{C}^3$ orbifolds, of ranks $p-q$ and $p+q$. Let us take the concrete example of the GLSM corresponding to $Y^{3,2}$, given by the charge matrix

$$Q = (1, 5, -3, -3) \tag{3.35}$$

The discussion in the preceding paragraph tells us that in the various classical limits of the Fayet-Iliopoulos parameter of this model, we recover, apart from flat space, the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold and two copies of the orbifold $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$.[4] The original GLSM charge matrix can now be enhanced by adding the twisted sectors corresponding to marginal deformations, and the full GLSM charge matrix is calculated to be

$$Q = \begin{pmatrix} 1 & 5 & -3 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 & -5 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 & -5 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{3.36}$$

The complete phase structure of the $Y^{3,2}$ space can now be obtained by analysing the Lagrangian corresponding to the charges of eq. (3.36) by making any combination of fields very large. Since the theory is supersymmetric, all the added twisted sector charges survive the GSO projection. This will in general not be the case for unstable spaces. For the charge

[4]The action of the $\mathbb{C}^3$ orbifold is $(Z_1, Z_2, Z_3) \to \left(\omega Z_1, \omega^2 Z_2, \omega^2 Z_3\right)$ with $\omega = e^{\frac{2\pi i}{5}}$ and that of the $\mathbb{C}^2$ orbifolds is $(Z_1, Z_2) \to \left(\omega' Z_1, \omega'^2 Z_2\right)$ with $\omega' = e^{\frac{2\pi i}{3}}$

matrix of eq. (3.36), we present the results for some of the phases of the theory. E.g if we make the fields $\rho_1, \rho_2, \rho_3, \rho_5$ and ρ_6 very large, the resultant flat sigma model metric is

$$\begin{aligned} ds^2 &= d\rho_4^2 + d\rho_7^2 + d\rho_8^2 + \rho_4^2 d(\theta_4 - \theta_3)^2 + \rho_7^2 d(\theta_7 - \theta_1 + 2\theta_2 + 3\theta_3 + \theta_5)^2 \\ &+ \rho_8^2 d(\theta_8 + 2\theta_1 - \theta_2 - \theta_3 + \theta_6)^2 \end{aligned} \tag{3.37}$$

Making the fields $\rho_2, \rho_4, \rho_6, \rho_7$ and ρ_8 acquire very large vevs, we obtain the sigma model metric

$$\begin{aligned} ds^2 &= d\rho_1^2 + d\rho_3^2 + d\rho_5^2 + \frac{\rho_1^2}{(2)^2} d(2\theta_1 - \theta_2 - \theta_4 + \theta_6 + \theta_8)^2 + \rho_3^2 d(\theta_3 - \theta_4)^2 \\ &+ \frac{\rho_5^2}{(2)^2} d(2\theta_5 + 3\theta_2 + 5\theta_4 + \theta_6 + 2\theta_7 + \theta_8)^2 \end{aligned} \tag{3.38}$$

which can be recognised to be the metric for $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, and arises in a limit of the supersymmetric $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ that we have seen earlier. Finally, say we look at the region of moduli space where the fields $\rho_1, \rho_2, \rho_5, \rho_6$ and ρ_7 acquire large vevs. In that case, the metric reads

$$\begin{aligned} ds^2 &= d\rho_3^2 + d\rho_4^2 + d\rho_8^2 + \frac{\rho_3^2}{(3)^2} d(3\theta_3 - \theta_1 + 2\theta_2 + \theta_5 + \theta_7)^2 \\ &+ \frac{\rho_4^2}{(3)^2} d(3\theta_4 - \theta_1 + 2\theta_2 + \theta_5 + \theta_7)^2 \\ &+ \frac{\rho_8^2}{(3)^2} d(3\theta_8 + 5\theta_1 - \theta_2 + \theta_5 + 3\theta_6 + \theta_7)^2 \end{aligned} \tag{3.39}$$

which is the metric for the orbifold $\mathbb{C}^3/\mathbb{Z}_3$.

The above analysis can be carried over to generic GLSMs representing unstable spaces. Let us concentrate on the class of GLSMs with charges

$$Q = (1, n_2, -n_3, -n_4) \tag{3.40}$$

where, without loss of generality we may take the first charge to be unity and we also assume that $n_4 > n_3 > n_2$, where the n_i are positive integers. This is an unstable conifold like singularity. Now take the case where n_2 acquires a large vev. The sigma model metric becomes,

$$ds^2 = \sum_{i=2}^{4} (d\rho_i)^2 + \frac{\rho_1^2}{n_2^2} d\left(n_2\theta_1 - \theta_2\right)^2 + \frac{\rho_3^2}{n_2^2} d\left(n_2\theta_3 + n_3\theta_2\right)^2 + \frac{\rho_4^2}{n_2^2} d\left(n_2\theta_4 + n_4\theta_2\right)^2 \tag{3.41}$$

This is recognised as the metric for the space $\mathbb{C}^3/\mathbb{Z}_{n_2}$ in the sense that when the Fayet-Iliopoulos parameter of the latter becomes very large, we recover the metric of eq. (3.41). [5] Now, we might add the twisted sectors corresponding to this orbifold, with the condition

[5] One might convert this charge matrix to standard form by making one of the integers to be unity using the fact that the integers in eq. (3.41) are defined modulo n_2, but that will not affect the physics.

that these are either relevant or marginal. [6] We find that including some of the twisted sectors in the analysis, our Lagrangian formulation predicts, in general, the existence of lower order conifold like singularities. The general example is less useful at this point of our discussion, and let us consider, as a concrete example, the GLSM of four fields with charges

$$Q = (1, 3, -5, -11) \tag{3.42}$$

The Lagrangian for this model is as usual given by setting $m = 4$ in eq. (2.5). We consider the various limits of the model by setting one field large at a time. Setting the vev of ϕ_1 to be very large, we recover flat space. A similar analysis for the Lagrangian with $\phi_i, i = 2, 3, 4$ having very large vevs gives rise to the sigma model metrics

$$\begin{aligned}
ds_2^2 &= d\rho_1^2 + d\rho_3^2 + d\rho_4^2 + \frac{\rho_1^2}{9}\left(3d\theta_1 + 2d\theta_2\right)^2 + \frac{\rho_3^2}{9}\left(3d\theta_3 + 2d\theta_2\right)^2 \\
&\quad + \frac{\rho_4^2}{9}\left(3d\theta_4 + 2d\theta_2\right)^2 \\
ds_3^2 &= d\rho_1^2 + d\rho_2^2 + d\rho_4^2 + \frac{\rho_1^2}{25}\left(d\theta_3 + 5d\theta_1\right)^2 + \frac{\rho_2^2}{25}\left(3d\theta_3 + 5d\theta_2\right)^2 \\
&\quad + \frac{\rho_4^2}{25}\left(4d\theta_3 + 5d\theta_4\right)^2 \\
ds_4^2 &= d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + \frac{\rho_1^2}{(11)^2}\left(d\theta_4 + 11d\theta_1\right)^2 \\
&\quad + \frac{\rho_2^2}{(11)^2}\left(3d\theta_4 + 11d\theta_2\right)^2 + \frac{\rho_2^2}{(11)^2}\left(6d\theta_4 + 11d\theta_3\right)^2
\end{aligned} \tag{3.43}$$

where the subscripts on the r.h.s indicates which field has been made large. These metrics are recognised to be the sigma model metrics for the orbifolds $\mathbb{C}^3/\mathbb{Z}_{3(2,2,2)}$ (or, equivalently, the supersymmetric $\mathbb{C}^3/\mathbb{Z}_{3(1,1,1)}$), $\mathbb{C}^3/\mathbb{Z}_{5(1,3,4)}$ and $\mathbb{C}^3/\mathbb{Z}_{11(1,3,6)}$ respectively. It is now clear how to enlarge the charge matrix. Including the relevant (and marginal) twisted sector states gives the enlarged charge matrix

$$Q = \begin{pmatrix} 1 & 3 & -5 & -11 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -3 & 0 & 0 & 0 \\ 1 & 3 & 6 & 0 & 0 & -11 & 0 & 0 \\ 2 & 6 & 1 & 0 & 0 & 0 & -11 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 & 0 & -11 \end{pmatrix} \tag{3.44}$$

and it is seen that apart from the third row, all other entries survive the type II GSO projection. Using our Lagrangian formulation, we can analyse the phases of this theory in full generality. E.g taking the truncated charge matrix (corresponding to the first two entries)

$$Q = \begin{pmatrix} 1 & 3 & -5 & -11 & 0 \\ 1 & 0 & 1 & 1 & -3 \end{pmatrix} \tag{3.45}$$

[6] It should be pointed out that in this analysis, we need to take care of the GSO projection of the twisted sectors. In general, a twisted sector will survive the type II GSO projection for $\sum_i Q_i =$ even. For the purpose of our analysis, we will broadly consider type 0 strings, it being understood that for type II theories, some of the twisted sectors are projected out.

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we see that making the first field very large, we get the GLSM (corresponding to an unbroken U(1)) with charge matrix $Q = (1, -2, -4, 1)$. Similarly, assigning a very large vev to the third field gives an unstable space represented by a U(1) GLSM with charges $(2, -2, -1, 5)$. The sigma model metrics corresponding to these can be easily written down.

The analysis with the full charge matrix is also simple. In this example, we get 42 distinct phases of the full GLSM, 22 of which corresponds to flat space, one each of $\mathbb{Z}_{11}$ and $\mathbb{Z}_5$ orbifolds, 2 each of $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds, 5 are $\mathbb{Z}_3$ and 9 are $\mathbb{Z}_2$ orbifolds. The exact action of these orbifolds can also be determined using the relevant Lagrangian in these phases. This illustrates the computational simplicity of our method of determining phases of generic GLSMs.

4. Conclusions

In this paper, we have extended the analysis of [9] to study the classical phases of generic GLSMs using the Lagrangian formulation. Our analysis gives a simple and powerful way of obtaining these phases, by tuning the fields which appear in the GLSM. We have shown how to construct the full phase structure of generic GLSMs, which might be unstable. To us, this completes the analysis initiated in [9], and our methods are complementary to those obtained in [8, 16]. However, there are certain issues that need to be examined.

Our formalism can be used to study the evolution of the D-brane boundary conditions for unstable spaces in the classical limits of the corresponding GLSMs, extending the preliminary analysis that we have presented here. As a simple example, the open string GLSM boundary condition $D_1\phi_i = 0$ (with the world sheet gauge field strength $v_{01} = 0$) [11] for the simplest fractional D2- brane (other fractional D2- branes are related to this by a quantum symmetry) in the unresolved orbifold phase can be seen from these formulae to translate into the simpler condition $\partial_1\phi_i = 0$ where the angular part of the ϕ_i now correspond to a gauge invariant angle. A similar result is obtained for other phases as well. It would be very interesting to do the corresponding analysis for generic unstable spaces using our Lagrangian formalism, especially for cases where there might be terminal singularities.

Further, having completely studied the full phase structure of a given GLSM, one might ask if the reverse engineering of singular spaces is possible. That is, given a certain number of orbifold singularities, is it possible to construct a GLSM that will have these orbifolds in their phases. An answer to this question will probably help us to have a better understanding of the D-brane quiver gauge theories corresponding to generic GLSMs, that have been analysed for the supersymmetric case in [7].

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